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Instability of laminar flows due to a film of adsorption

By CHIA-SHUN YIH
Department of Engineering Mechanics,
The University of Michigan, Ann Arbor, Michigan

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When a horizontal layer of viscous liquid with an adsorption film of varying concentration as its upper boundary is set in motion by a steady translation of its lower boundary, plane Couette flow with zero surface velocity is possible. In this paper the stability of this flow is considered, and it is found that the liquid layer can be unstable for long waves. The instability found for this flow, however, exists also for other laminar flows with an adsorbed film, and plane Couette flow has been chosen only as a simple means of demonstration.

1. Introduction

When surface-active material is present in an adsorbed film at the surface of a liquid, the surface tension of the liquid may change from place to place, giving rise to non-zero shear stresses at the surface. If the surface concentration of the adsorbed material is denoted by γ, the surface diffusivity by D, and the time by t, the equation of continuity for the adsorbed material is (Levich 1962, p. 393)

$$\partial\gamma/\partial t + \mathbf{div}(\gamma\mathbf{v} - D\,\mathbf{grad}\,\gamma) + j_n = 0, \tag{1}$$

in which $\mathbf{v}$ is the velocity of the fluid at the surface, j_n is the flux of the material from the surface to the interior of the liquid, per unit time and per unit area, and the divergence applies to the surface only. The normal flux j_n is usually assumed negligible compared with the other terms. We shall make the same assumption here and shall furthermore assume D to be constant. Thus the preceding equation will be written as

$$\partial\gamma/\partial t + \mathbf{div}\,(\gamma\mathbf{v}) = D\Delta\gamma, \tag{2}$$

Δ being the Laplacian operator.

Landau & Lifshitz (1959, pp. 242-3) presented a solution for a flow in a deep channel joining two reservoirs and driven by surface shear arising from the non-homogeneity of γ on the surface. Instead of (2) they presented its non-diffusive form

$$\partial\gamma/\partial t + \mathbf{div}\,(\gamma\mathbf{v}) = 0; \tag{2a}$$

but the solution given does not satisfy (2*a*) or (2) and is therefore not valid. Landau & Lifshitz attributed the solution to Levich, and gave reference to the Moscow edition (1952) of the latter's book *Physicochemical Hydrodynamics*. The only edition of that book at the present writer's disposal is the second edition in its English version (Levich 1962). When the writer searched the second edition he could not find the solution Landau & Lifshitz referred to. It is not clear whether

Levich realized his solution was wrong and therefore withdrew it in the second edition of his book, or he merely suppressed it for other reasons. At any rate any interested reader can demonstrate to his own satisfaction that the solution quoted by Landau & Lifshitz is incorrect. It turns out that if one insists on giving a valid solution of Levich's original problem one must take into consideration the longitudinal variation of velocity and the surface height. If one still wants to have a simple and truly one-dimensional solution, one must consider a different problem, in which the free surface must be stationary, in order that (2) can be satisfied. Since in a one-dimensional flow the surface must be flat, and since on that flat surface the pressure must be constant, the flow can either be a horizontal plane Couette flow with the lower boundary moving, or a plane Poiseuille-Couette flow, with the lower boundary inclined to the horizontal but not necessarily moving.

For simplicity we consider the former flow in this paper, and investigate its stability. It will be seen that this flow can be unstable for long waves. The motion of the lower boundary does make the flow rather special, and this speciality is of course not attractive. But the instability to be demonstrated no doubt exists also for more natural free-surface flows with an adsorbed film, such as (*a*) flow of a liquid layer down an inclined plane, or (*b*) the nearly parallel flow of a liquid layer on a horizontal bottom, with a nearly parabolic velocity distribution in each section due to a longitudinal pressure gradient which is in turn due to the slope of the free surface induced by the motion of the surface film. We wish to demonstrate the kind of instability which can occur for flows with an adsorbed film, and have chosen plane Couette flow merely as a simple vehicle for demonstration.

2. Primary flow

Consider a unidirectional steady flow in the X-direction of a layer of viscous liquid of depth d. The velocity, denoted by $\overline{u}$, is a function of Y only. The lower boundary moves with a constant speed V (see figure 1). We shall consider only the case $\overline{u}(0) = 0$. Since the flow is unsteady, the solution of (2) is, with $\overline{\gamma}$ denoting the γ for the primary flow,

$$\overline{\gamma} = \gamma_0 + \gamma_1 X, \tag{3}$$

in which γ_0 is the value of $\overline{\gamma}$ at the origin, and

$$\gamma_1 = [\overline{\gamma}(L) - \overline{\gamma}(-L)]/2L. \tag{4}$$

The length of the channel is $2L$ and supposed to be very large compared with d. The reservoir with greater concentration of the adsorbed material is situated at $X = L$, and the other reservoir at $x = -L$. Note that, if $\overline{u}(0)$ is not zero, (3) is not a solution of (2). This is the reason for demanding zero velocity at the surface.

If the surface tension is denoted by T, the shear stress on the surface, where $Y = 0$, is, for the co-ordinates chosen in agreement with an earlier work (Yih 1963),

$$\tau_{21} = -\partial T/\partial X = \delta\, \partial\overline{\gamma}/\partial X = \delta\gamma_1, \tag{5}$$

in which

$$-\delta = \partial T/\partial \gamma \tag{6}$$

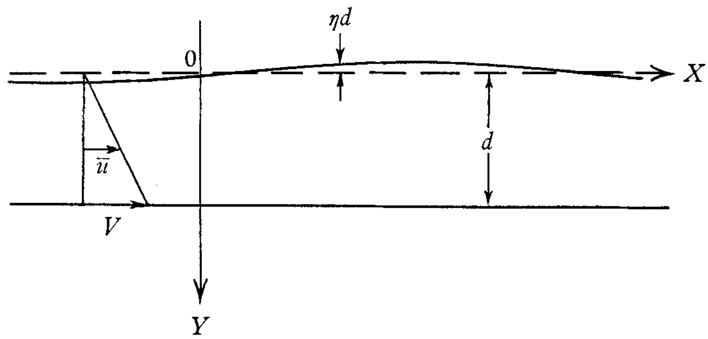

FIGURE 1. Definition sketch.

is supposed to be constant. It is obvious that the flow due to a moving lower boundary in the presence of a stationary upper surface at constant pressure must be a plane Couette flow described by

$$U = y, \tag{7}$$

in which

$$U = \bar{u}/V, \quad y = Y/d. \tag{8}$$

However (5) must be satisfied. Hence, with μ denoting the viscosity,

$$\mu\, d\bar{u}/d\gamma = \gamma_1 \delta. \tag{9}$$

This implies

$$\delta\gamma_1 d/\mu V = 1. \tag{10}$$

Equations (3) and (7), with the restriction (10), describe the plane Couette flow under consideration. The pressure gradient is of course zero, so that, if $\bar{p}$ denotes the pressure of the primary flow,

$$\partial\bar{p}/\partial X = 0. \tag{11}$$

3. General formulation of the stability problem

We consider, as usual, only two-dimensional disturbances. With u and v denoting the velocity components in the directions of increasing X and Y, respectively, and with p indicating the pressure, the Navier–Stokes equations are

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial X} + v\frac{\partial u}{\partial Y} = -\frac{1}{\rho}\frac{\partial p}{\partial X} + \nu\Delta u, \tag{12}$$

$$\frac{\partial v}{\partial t} + u\frac{\partial v}{\partial X} + v\frac{\partial v}{\partial Y} = -\frac{1}{\rho}\frac{\partial p}{\partial Y} + g + \nu\Delta v, \tag{13}$$

in which ν is the kinematic viscosity. The co-ordinates X and Y are defined in figure 1. The equation of continuity is

$$\partial u/\partial X + \partial v/\partial Y = 0. \tag{14}$$

The substitutions

$$\left.\begin{aligned} (u_1, v_1) = (u, v)/V, \quad (x, y) = (X, Y)/d, \\ p_1 = p/\rho V^2 \quad \text{and} \quad \tau = tV/d, \end{aligned}\right\} \tag{15}$$

can be used to reduce (12), (13) and (14) to their dimensionless forms:

$$\frac{\partial u_1}{\partial \tau}+u_1\frac{\partial u_1}{\partial x}+v_1\frac{\partial u_1}{\partial y} = -\frac{\partial p_1}{\partial x}+\frac{1}{R}\Delta u_1, \tag{16}$$

$$\frac{\partial v_1}{\partial \tau}+u_1\frac{\partial v_1}{\partial x}+v_1\frac{\partial v_1}{\partial y} = -\frac{\partial p_1}{\partial y}+F^{-2}+\frac{1}{R}\Delta v_1, \tag{17}$$

$$\frac{\partial u_1}{\partial x}+\frac{\partial v_1}{\partial y} = 0, \tag{18}$$

in which Δ is now in terms of x and y, and

$$R = Vd/\nu \quad \text{and} \quad F = V(gd)^{-\frac{1}{2}} \tag{19}$$

are the Reynolds number and Froude number, respectively.

Let $$u_1 = U+u', \quad v_1 = v', \quad \text{and} \quad p = P+p', \tag{20}$$

in which U is the dimensionless velocity of the primary flow given by (7), P is the dimensionless pressure of the primary flow, and the accented quantities are perturbation quantities of magnitude very much smaller than unity. Substituting (20) into (16), (17) and (18), subtracting out the terms representing the primary flow only and neglecting quadratic terms in the perturbation quantities, we have

$$u'_\tau+Uu'_x+U_y v' = -p'_x+R^{-1}\Delta u', \tag{21}$$

$$v'_\tau+Uv'_x = -p_y+R^{-1}\Delta v', \tag{22}$$

$$u'_x+v'_y = 0, \tag{23}$$

with subscripts denoting partial differentiation. Equation (23) permits the use of a stream function ψ, in terms of which

$$u' = \psi_y, \quad v' = -\psi_x. \tag{24}$$

Equations (21) and (22) can be written as

$$\psi_{y\tau}+U\psi_{xy}-U_y\psi_x = -p'_x+R^{-1}\Delta\psi_y, \tag{25}$$

$$\psi_{x\tau}+U\psi_{xx} = p'_y+R^{-1}\Delta\psi_x. \tag{26}$$

The boundary conditions at the bottom are

$$\text{(i)}\ u' = \psi_y = 0 \quad \text{and} \quad \text{(ii)}\ v' = -\psi_x = 0.$$

Before we can formulate the boundary conditions at the free surface an equation governing the perturbation in γ is needed. With

$$\gamma = \bar{\gamma}+\gamma_0\gamma', \tag{27}$$

in which γ' is the dimensionless perturbation quantity in γ, the linearized and dimensionless form of (1) is

$$\frac{\partial \gamma'}{\partial \tau}+\frac{u'\gamma_1 d}{\gamma_0}+\frac{\bar{\gamma}}{\gamma_0}\frac{\partial u'}{\partial x} = \frac{1}{P\acute{e}}\Delta\gamma', \tag{28}$$

in which
$$Pé = Vd/D, \tag{29}$$

and Δ is in terms of x and y. In obtaining (28), it is understood that the curvilinear distance along the free surface can be identified with x, since the amplitude of the waves under consideration is supposed small. Equation (28) can be written as

$$\gamma'_\tau + \frac{\gamma_1 d}{\gamma_0}\psi_y + \frac{\overline{\gamma}}{\gamma_0}\psi_{xy} = \frac{1}{Pé}\Delta\gamma'. \tag{28a}$$

The boundary condition *at the free surface* regarding the shear stress is then

$$\mu\left(\frac{\partial u}{\partial Y} + \frac{\partial v}{\partial X}\right) = \delta\gamma_1 + \delta\gamma_0\frac{\partial \gamma'}{\partial X},$$

or, in dimensionless terms,

$$\frac{\partial u_1}{\partial y} + \frac{\partial v_1}{\partial x} = \frac{d\delta\gamma_1}{\mu V} + \frac{\delta\gamma_0}{\mu V}\gamma'_x. \tag{30}$$

Since (30) is, *a priori*, to be applied at the free surface rather than at $y = 0$, its final form is, after (20), (5), (10) and (24) have been utilized,

$$\text{(iii)} \quad \frac{d^2U}{dy^2}\eta + \psi_{yy} - \psi_{xx} = \frac{\gamma_0}{\gamma_1 d}\gamma'_x,$$

in which η is the dimensionless displacement of the free surface, as shown in figure 1. Equation (30) can now be applied at $y = 0$. Of course, for plane Couette flow, the first term is zero, and it would not have made any difference if (30) had been applied at $y = 0$ to start with.

The normal-stress condition at the free surface is

$$\left(-p_1 + \frac{2}{R}\frac{\partial v_1}{\partial y}\right)\rho V^2 + T\frac{\partial^2(\eta d)}{\partial X^2} = 0,$$

or
$$-p_1 + \frac{2}{R}\frac{\partial v_1}{\partial y} + S\frac{\partial^2 \eta}{\partial x^2} = 0, \quad S = \frac{T}{\rho d V^2}.$$

Since this has to be applied at the free surface, and not merely at $y = 0$, it can be written further in the form

$$-P - P_y\eta - p' - 2R^{-1}\psi_{xy} + S\eta_{xx} = 0.$$

Now $P(0) = 0$ and $P_y(0) = F^{-2}$, so that the normal-stress condition can be written as

$$\text{(iv)} \quad F^{-2}\eta + p' + 2R^{-1}\psi_{xy} - S\eta_{xx} = 0.$$

The conditions (iii) and (iv) all involve η. To determine η in terms of ψ, we use the kinematic condition

$$-\psi_x = v' = \eta_\tau. \tag{31}$$

We now consider a spatially growing or damping disturbance of (dimensionless) angular frequency ω, and assume

$$(\psi, p', \gamma') = [\phi(y), f(y), \chi(y)]\exp i\left[\int \alpha\, dx - \omega\tau\right], \tag{32}$$

in which α is a function of x. The reason for not assuming α to be independent of x is that the surface tension is not constant, and therefore the diffusion equation of the surface material and one free-surface boundary condition have x-dependent coefficients. We shall retain the x-dependence of these coefficients, and shall consider the instability of the disturbance when α and ω are both small. Since

$$\alpha = \alpha_r + i\alpha_i,$$

the flow is unstable or stable according as α_i is positive or negative, provided the waves propagate upstream† (toward decreasing values of x). Equation (31) then assumes the form

$$\eta = \frac{\alpha\phi(0)}{\omega} \exp i\left[\int \alpha\, dx - \omega\tau\right]. \tag{33}$$

The equations of motion become

$$i(\alpha U - \omega)\,\phi' - i\alpha U'\phi = -i\alpha f + \mathrm{R}^{-1}(\phi''' - \alpha^2\phi'), \tag{34}$$

$$\alpha(\omega - \alpha U)\,\phi = f' + i\alpha R^{-1}(\phi'' - \alpha^2\phi), \tag{35}$$

and (28*a*) becomes

$$-i\omega\chi + \frac{\gamma_1 d}{\gamma_0}\phi' + \frac{i\alpha\bar{\gamma}}{\gamma_0}\phi' = -\frac{\alpha^2}{P\acute{e}}\chi, \tag{36}$$

in which $\bar{\gamma}$ is not constant but a linear function of x. In (34), (35) and (36), the primes denote differentiation with respect to y. Elimination of f between (34) and (35) produces the Orr–Sommerfeld equation

$$\phi^{\mathrm{iv}} - 2\alpha^2\phi'' + \alpha^4\phi = iR[(\alpha U - \omega)(\phi'' - \alpha^2\phi) - \alpha U''\phi]. \tag{37}$$

We shall now write the boundary conditions in terms of ϕ and χ. These are

(i) $\phi'(1) = 0$,

(ii) $\phi(1) = 0$,

(iii) $\phi''(0) + \alpha^2\phi(0) = (i\alpha\gamma_0/\gamma_1 d)\,\chi(0)$,

(iv) $[\alpha^2(RF^{-2} + \alpha^2 SR)/\omega]\,\phi(0) + \alpha R U'(0)\,\phi(0) + (R\omega + 3\alpha^2 i)\,\phi'(0) - i\phi'''(0) = 0$.

In obtaining the final form of (iv), p' has been evaluated from (32), with f given by (34), and with $U(0) = 0$.

The formulation is now complete. We should note that, since T is not constant for all x, nor is S. Since S is associated with α^3 in (iv), we can, to the stage of approximation achieved in this paper, ignore it, and write the last boundary condition as

$$\text{(iv)}\quad R[\alpha/F^2\omega + U'(0)]\,\phi(0) + (R\omega + 3a^2 i)\,\phi'(0) - i\phi'''(0) = 0.$$

The x-dependence of S, however, should be retained in higher-order approximations.

† As will be shown, the waves treated in this paper propagate upstream. In case the waves propagate downstream, the flow is unstable if α_i is negative, and stable if α_i is positive.

4. Solution of the stability problem for plane Couette flow

In solving the differential system governing stability formulated in the last section, we shall adopt the procedure in Yih (1963). Since $U'' = 0$ in the present case, the first approximation is governed by the equations

$$\phi_0^{\text{iv}} = 0, \tag{37a}$$

$$-i\omega\chi_0 + (\gamma_1 d/\gamma_0)\,\phi_0' = 0, \tag{36a}$$

and the boundary conditions

(i) $\phi_0'(1) = 0$, (ii) $\phi_0(1) = 0$,

(iii) $\phi_0''(0) = (i\alpha_0\gamma_0/\gamma_1 d)\,\chi_0(0)$, (iv) $\phi_0'''(0) = 0$.

Note that $\alpha\chi$ or $\omega\chi$ is of the same order as ϕ and its derivatives, and hence must be kept in (36a) and (iii). Combining these two equations, we have

$$\phi_0''(0) = \omega^{-1}\alpha_0\,\phi_0'(0). \tag{38}$$

The solution of (37a), with conditions (i), (ii), (iv) and (38), is straightforward and is

$$\phi_0 = (1-y)^2, \quad \alpha_0 = -\omega. \tag{39}$$

Thus the waves propagate in the negative x-direction with dimensionless speed 1, or dimensional speed V.

The next approximation involves the equation

$$\phi_1^{\text{iv}} = -i2\omega R(1+y), \tag{40}$$

whose solution is†

$$\phi_1 = -i2\omega R(\tfrac{1}{24}y^4 + \tfrac{1}{120}y^5) + \Delta B y + \Delta C y^2 + \Delta D y^3.$$

Two of the boundary conditions for ϕ_1 are

(i) $\phi_1'(1) = 0$, (ii) $\phi_1(1) = 0$.

Condition (iv) now has the form

(iv) $-\omega R(3-F^{-2}) - i\phi_1'''(0) = 0$.

As to the equation corresponding to (38), (36) and the original form of (iii) at this stage give it the form

$$\phi_1'' + \phi_1' = 2\left(\frac{\Delta\alpha}{\alpha_0} - \frac{i\omega}{P\acute{e}} - \frac{i\omega\overline{\gamma}}{\gamma_1 d}\right). \tag{41}$$

Thus

$$\Delta B + \Delta C + \Delta D = \tfrac{1}{10}i\omega R, \quad \Delta B + 2\Delta C + 3\Delta D = \tfrac{1}{12}i5\omega R, \quad 6\Delta D = i\omega R(3-F^{-2}),$$

$$\Delta B + 2\Delta C = 2\left(\frac{\Delta\alpha}{\alpha_0} - \frac{i\omega}{P\acute{e}} - \frac{i\omega\overline{\gamma}}{\gamma_1 d}\right),$$

from which (since $\alpha_0 = -\omega$)

$$\Delta\alpha = i\omega^2 R\left(\frac{13}{24} - \frac{1}{RP\acute{e}} - \frac{1}{4F^2} - \frac{\overline{\gamma}}{\gamma_1 dR}\right). \tag{42}$$

† See Yih (1963) for the reason for not providing a term ΔA.

By utilizing (10) one can interpret (42) in the following dimensional terms: if

$$\frac{gd}{4}+\frac{D\nu}{d^2}+\frac{\delta\overline{\gamma}}{\rho d}<\frac{13}{24}\left(\frac{\delta\gamma_1 d}{\mu}\right)^2, \tag{43}$$

the flow is unstable. This criterion is x-dependent. The surface-active agent is placed at $x = L$, and therefore $\overline{\gamma}$ increases with x. Thus, if the flow is neutral for one value of x, it is unstable for all algebraically smaller values of x, and the degree of instability increases as x decreases.

For higher approximations ΔB, ΔC, and ΔD must be evaluated. These are functions of x.

In (43), the first term represents the stabilizing effect of gravity, and the second the stabilizing effect of the diffusivity of the material and of viscosity. The third, having its origin in the third term in (36) and eventually in the third term in (28*a*), represents the stabilizing effect of the *stretching* of the film. The right-hand side of (43) represents the destabilizing effect of the gradient of surface tension. It is equal to $13V^2/24$, in which V is the necessary bottom velocity to make the surface velocity of the primary flow equal to zero, and thus to make truly parallel flow possible. We may also consider it to represent the destabilizing effect of inertia, since it has its origin in the inertial terms on the right-hand side of (40). But these inertial terms in turn owe their origin to the primary flow, which is intimately related to the surface-tension gradient.

Finally, we may wonder whether the inequality (43) for instability is ever satisfied in a realistic situation. It is not unrealistic to consider a layer of water of depth 0·02 ft. flowing from one reservoir to another 1 ft. away, where the surface contamination reduces the surface tension to one half of its value without such contamination. The surface tension of water at 70 °F is 0·005 lb./ft. Thus $\delta\overline{\gamma}$ (δ assumed constant) has one half of that value at the surface of the contaminated reservoir. The quantity $\delta\gamma_1 \times 1$ ft. also has the value 0·0025 lb./ft. Since $\rho = 1{\cdot}94$ slug/ft.^{3}, $g = 32{\cdot}2$ ft./sec^2, $\nu = 1{\cdot}05\times10^{-5}$ ft.^{2}/sec, $\mu = 2{\cdot}04\times10^{-5}$ lb.-sec/ft.^{2}, and D can be assumed to be of the same order as ν and will be assumed equal to 10ν on the safe side, we have, in units of ft.^{2}/sec^2,

$$\tfrac{1}{4}gd = 0{\cdot}160, \quad D\nu/d^2 = 2{\cdot}75\times10^{-6},$$
$$\delta\overline{\gamma}/\rho d = 0{\cdot}013, \quad \tfrac{13}{24}(\delta\gamma_1 d/\mu)^2 = \tfrac{13}{4}.$$

Thus (43) is definitely satisfied. It is also evident that, of the three stabilizing effects, that due to gravity is the most important.

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